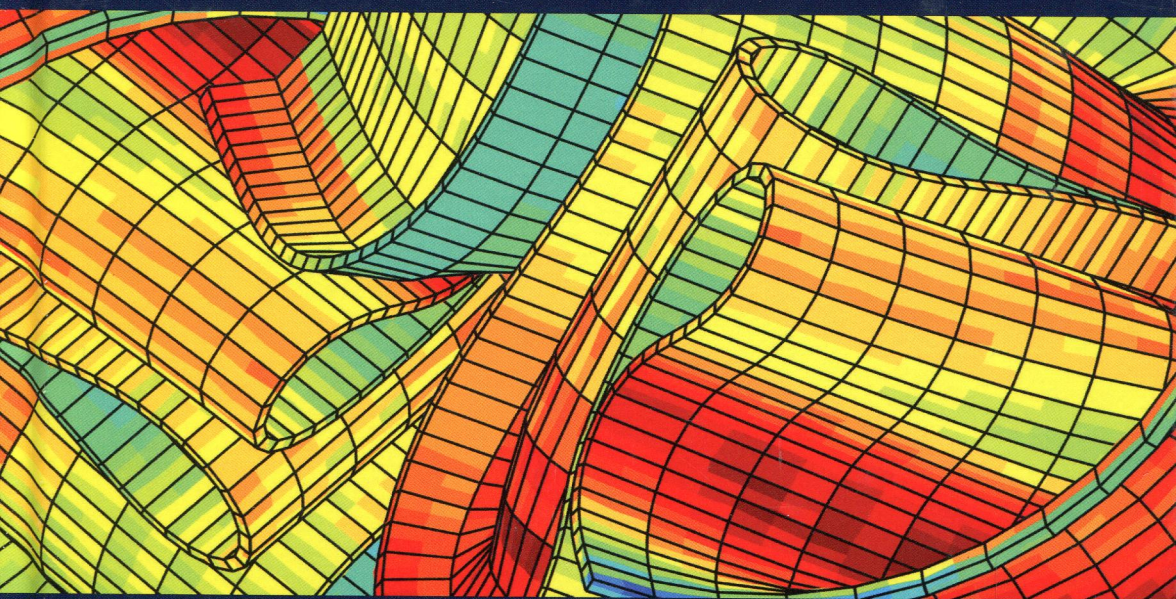


NUMERICAL METHODS IN ENGINEERING SERIES



# Numerical Methods in Contact Mechanics

**Vladislav A. Yastrebov**

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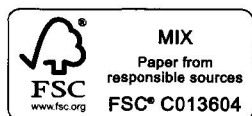
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